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Remarks on the Thickness and Outerthickness of a Graph

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Abstract—The *thickness* of a graph is the minimum number of planar subgraphs into which the graph can be decomposed. The thickness of complete bipartite graphs $K_{m,n}$ is known for almost all values of m and n. In this paper, we solve the thickness of complete bipartite graphs for unknown cases $m < 30$, $m \leq n$. The new solutions coincide with the general formula and they were obtained by using a simulated annealing algorithm.

The *outerthickness* of a graph is the minimum number of outerplanar subgraphs into which the graph can be decomposed. We give lower and upper bounds for outerthickness in the terms of the minimum and maximum degree of a graph. Let δ be the minimum degree and Δ the maximum degree of a graph G. We show that the following bounds hold for outerthickness: $\lceil \delta/4 \rceil \leq \Theta_o(G) \leq \lceil \Delta/2 \rceil$. We also discuss the possibility of determining the upper bound for outerthickness using the number of edges of a given graph.

Keywords—Combinatorial problems, Graph thickness, Graph outerthickness.

1. INTRODUCTION

The *thickness* of a graph G, denoted $\Theta(G)$, is the minimum number of planar subgraphs into which the graph can be decomposed. Thickness is one of the topological invariants that describe a graph's embeddability into a plane. Another topological invariant is *outerthickness*, denoted $\Theta_o(G)$, which is the minimum number of outerplanar subgraphs into which the graph can be decomposed.

Determining the maximum outerplanar subgraph of a given graph is known to be an NP-complete problem [1], but the complexity status of outerthickness is open. Since the thickness problem is NP-complete [2], it is plausible to conjecture that also determining the outerthickness of a graph is NP-complete.

Thickness is known for almost all complete bipartite graphs [3]. In a survey on biplanar graphs, Beineke [4] writes that "To the best of our knowledge, no progress has been made on this problem in the past 30 years".

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In this paper, we determine the thickness of complete bipartite graphs $K_{m,n}$ for all unknown values $m < 30$. Solutions have been obtained by using a simulated annealing algorithm [5]. Outerthickness is known for all complete bipartite graphs [6].

Wessel showed that if δ and Δ are the minimum and maximum degree of a graph G, then $\lceil(\delta+1)/6\rceil \leq \Theta(G) \leq \lceil\Delta/2\rceil$ [7]. Halton gave later independently similar results [8]. The upper bound was proven to be tight by Sýkora *et al.* [9]. Another approach is based on the total number of edges. Dean *et al.* showed that $\Theta(G) \leq \lfloor\sqrt{e/3}+3/2\rfloor$ [10] for a graph with e edges.

In this paper, we give similar results for determining the lower and upper bounds for outerthickness. We show that

$$\lceil\delta/4\rceil \leq \Theta_o(G) \leq \lceil\Delta/2\rceil .$$

Hence, we strengthen the previous results by showing that the upper bound known to hold for the thickness of a graph hold also for outerthickness. The lower bound for the outerthickness is new.

The rest of this note is organized as follows. In Section 2, we introduce the solutions for complete bipartite graphs $K_{m,n}$, $m \leq n$, for all $m < 30$. A sample decomposition is given in the Appendix. In Section 3, we give upper and lower bounds for outerthickness. We apply straightforward proof techniques introduced by Halton [8] and Wessel [7]. In Section 4, we discuss the possibility of using proof technique introduced by Dean *et al.* [10] for determining the upper bound for the outerthickness of a graph and give a conjecture on outerthickness as a function of the number of edges. For the basic graph-theoretical concepts we refer to [11]. In what follows, we assume that graphs are simple, that is, no self-loops or multiple edges are allowed.

2. THICKNESS OF COMPLETE BIPARTITE GRAPHS

Beineke *et al.* determined the thickness of complete bipartite graphs $K_{m,n}$ for almost all values of m and n.

THEOREM 2.1. *(See [3].) For complete bipartite graphs,*

$$\Theta\left(K_{m,n}\right) = \left\lceil \frac{mn}{2\left(m+n-2\right)} \right\rceil ,$$

with $m \leq n$, except possibly when m and n are odd, and there exists an integer k satisfying

$$n = \left\lfloor \frac{2k\left(m-2\right)}{m-2k} \right\rfloor .$$

For example, it was unknown if $\Theta(K_{17,21})$ is equal to 5 or 6, the thickness of $K_{17,21}$ is at least 5 due to Euler's polyhedron formula and it cannot be more than $\Theta(K_{18,21}) = 6$ or $\Theta(K_{17,22}) = 6$. In general, the unknown values of $\Theta(K_{m,n})$ are quite rare. For an arbitrary m, there are fewer than $m/4$ unsolved cases [4]. In Table 1, we have listed the first 11 pairs of m and n with unknown thickness $(K_{m,n})^1$, the number of edges (mn) and the solution obtained by using a simulated annealing algorithm by Poranen [5] ($\Theta(K_{m,n})$). The algorithm is based on dividing first an input graph into planar subgraphs. Then, the algorithm tries to decrease the number of subgraphs by moving and swapping edges between different subsets. This process is guided by the simulated annealing optimization scheme.

The computational results strongly support that the general formula is correct for all unknown cases, since the exceptions were most likely to occur for small values [12].

Our results verify that the general formula for the thickness of complete bipartite graphs $K_{m,n}$, $m \leq n$ holds for all $m < 30$. The thickness of $K_{13,17}$ was actually already solved by Beineke *et al.* [3] as a special case The planar decomposition of $K_{17,21}$ into five subgraphs is given in the

[1] All optimal planar decompositions can be downloaded from `http://www.cs.uta.fi/~tp/apptopinv`.

Table 1. Thickness in some previously unknown cases

$K_{m,n}$	mn	$\Theta(K_{m,n})$
$K_{13,17}$	221	4
$K_{17,21}$	357	5
$K_{19,29}$	551	6
$K_{19,47}$	893	7
$K_{21,25}$	525	6
$K_{23,75}$	1725	9
$K_{25,29}$	725	7
$K_{25,59}$	1475	9
$K_{27,71}$	1917	10
$K_{29,33}$	957	8
$K_{29,129}$	3741	12

Appendix. Vertices $0 \dots 16$ belong to the first vertex set and vertices $17 \dots 37$ to the second set. The subgraphs were drawn using AGD [13] and the drawings were improved manually.

3. BOUNDS FOR OUTERTHICKNESS

An outerplanar graph is maximal if no edge can be added without losing outerplanarity. Any triangulation of a polygon constructs a maximal outerplanar graph. Using Euler's polyhedron formula, it is straightforward to find the number of edges in a maximal outerplanar graph.

LEMMA 3.1. *(See [11].) Let G be a maximal outerplanar graph with n vertices. Then, G has $2n-3$ edges.*

Wessel and Halton [7,8] showed that any graph G with minimum degree δ and maximum degree Δ has $\lceil(\delta+1)/6\rceil \le \Theta(G) \le \lceil\Delta/2\rceil$. In order to give similar results for outerplanar graphs, we recall some preliminary results from Halton's article that imply the claimed upper bound. Lemma 3 2 describes the possibility to augment a given graph to a regular graph containing the original graph as a subgraph.

LEMMA 3.2. *(See [8].) Given a graph G of degree Δ, there exists a regular graph of degree Δ containing G as a subgraph*

A graph is two-factorable, if it is a union of edge-disjoint cycles. The well-known Petersen's theorem states that regular graphs of even degree have this property.

LEMMA 3.3. *(See [14]) If a graph is regular and of even degree, then it is two-factorable.*

Next, we give lower and upper bounds for outerthickness in the terms of minimum and maximum degree. The proof of the following theorem is similar to that given in [7,8], but we use outerplanar subgraphs instead of planar subgraphs.

THEOREM 3.1. *Let G be a graph with minimum degree δ and maximum degree Δ. Then, $\lceil\delta/4\rceil \le \Theta_o(G) \le \lceil\Delta/2\rceil$.*

PROOF. Graph G has at least $\delta n/2$ edges. Since a maximal outerplanar graph has at most $2n-3$ edges by Lemma 3 1, we have

$$\Theta_o(G) \ge \left\lceil \frac{\delta n}{2(2n-3)} \right\rceil \ge \left\lceil \frac{\delta n}{4n} \right\rceil = \left\lceil \frac{\delta}{4} \right\rceil .$$

By Lemma 3.2, there exists a regular graph G_r of degree Δ, if Δ is even, or of degree $\Delta+1$, if Δ is odd, containing G as a subgraph. Applying Lemma 3.3 to extract cycles (which are obviously outerplanar) from the regular graph of even degree and using induction over the degree of the regular graph, the claimed upper bound can be obtained. The theorem follows.

For more details concerning the proof of the second inequality, see Halton's article [8]. Since $\Theta_o(G) \geq \Theta(G)$ and the upper bound is tight for thickness [9], it follows that the upper bound is tight also for outerthickness.

4. AN UPPER BOUND CONJECTURE FOR OUTERTHICKNESS

The *arboricity* of a graph is the minimum number of acyclic graphs into which the edges of G can be decomposed. Since the arboricity of a graph with e edges is at most $\lceil\sqrt{e/2}\rceil$ [10], the same bound holds also for the outerthickness of a graph.

If the proof technique of Dean *et al.* [10] is applied straightforward to outerplanar graphs, the bound $\lceil\sqrt{e/2}+1/2\rceil$ is obtained.

The upper bound above is of the right order, since $\Theta_o(K_n) = \lceil n+1/4\rceil$ when $n > 7$ [10]. On the other hand, $\Theta_o(K_n)$ is approximately $\sqrt{e/8}$ and $\Theta_o(K_{n,n})$ is approximately $\sqrt{e/9}$. It seems that the constant is not the best possible. We close this note by giving the following conjecture.

CONJECTURE 4.1. $\Theta_o(G) \leq \sqrt{e/8}+O(1)$ for an arbitrary graph G.

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APPENDIX

DECOMPOSITION OF $K_{17,21}$ INTO FIVE PLANAR GRAPHS

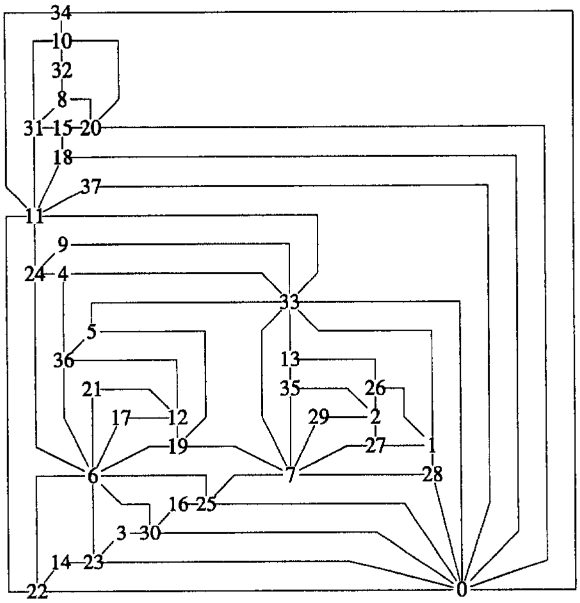

Figure 1

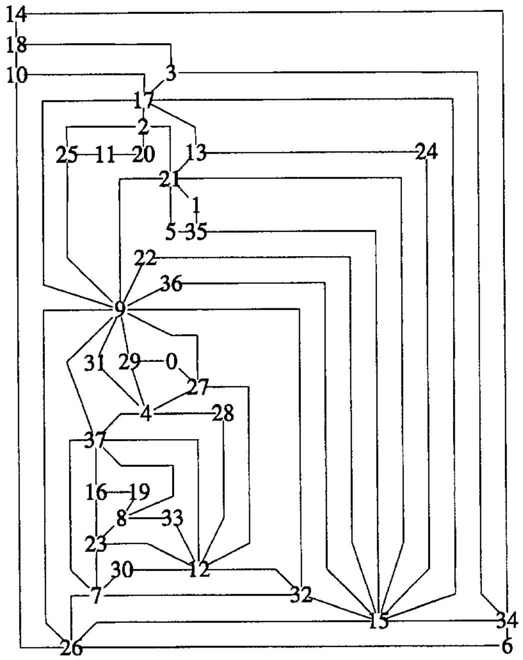

Figure 2.

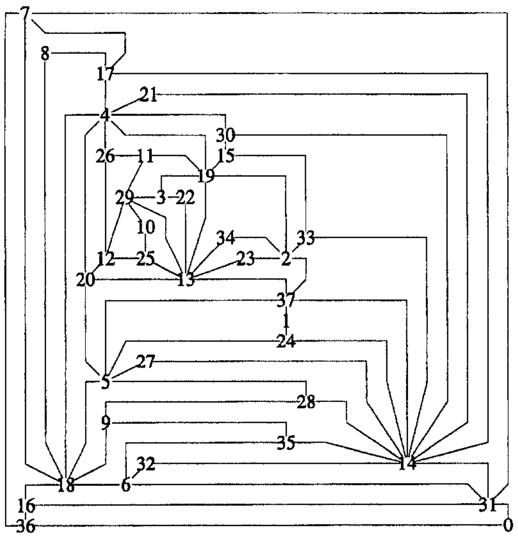

Figure 3

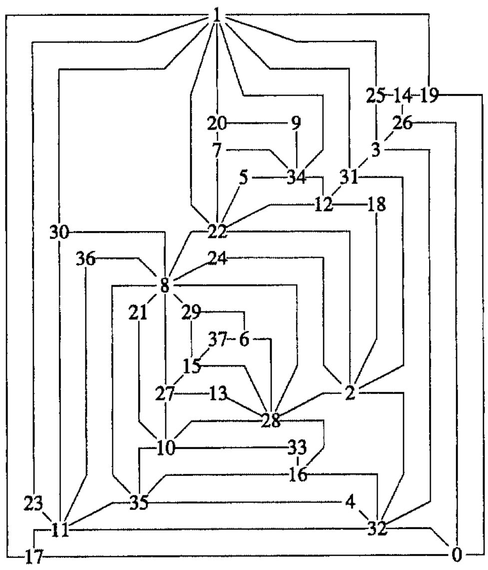

Fıgure 4.

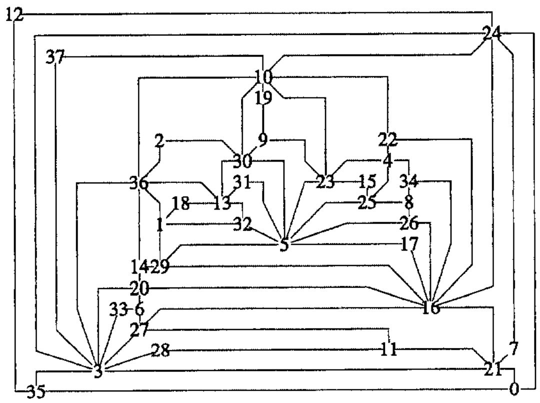

Fıgure 5